Lecture 2: Stresses in Pavements
Stresses in Layered Systems

- At any point, 9 stresses exist. They are 3 normal stresses ($\sigma_z$, $\sigma_r$, $\sigma_t$) and 6 shearing stresses ($\tau_{rz} = \tau_{zr}$, $\tau_{rt} = \tau_{tr}$, and $\tau_{tz} = \tau_{zt}$).
- At each point in the system there exists a certain orientation of the element such that the shearing stresses acting on each face are zero.
- The normal stresses under this condition are principal stresses and are denoted by $\sigma_1$ (major), $\sigma_2$ (intermediate) and $\sigma_3$ (minor) principal stresses.
Layered Elastic Models

- **Boussinesq Theory**
  - Layer characterized by $E$ and $v$.
  - Homogeneous, isotropic, stresses and strains determined from a point load on surface of semi-infinite half-space.
  - Further work expanded solutions to uniform, circular load by means of integration over the loaded area.

- **Burmister Theory**
  - Two-layered system developed in 1940’s.

- **Three Layer System**
  - Developed by Acum and Fox in 1950’s.
  - Homogeneous, isotropic, full friction between layers, stresses on centerline only.
  - Continuity equation satisfied at interfaces (same vertical stress, shear stress, vertical displacement, and radial displacement).
  - If frictionless, shear stress and radial displacement are zero at interface.
Layered Elastic Models (LEM)

**Basic Assumptions:**

- Each layer is homogeneous, isotropic, and linearly elastic with an elastic modulus of $E$ and Poisson's ratio of $\nu$.
- The material is weightless (geostatic stresses are ignored).
- Each layer has a finite thickness, except the lowest layer.
- A uniform pressure is applied over a circular area.
- Interface condition (continuity vs. frictionless).
- Surface shear forces are not present.
Flexible Pavement Responses

- Responses in flexible pavements under the moving traffic loads are greatly influenced by:
  - Type of flexible pavement.
  - Ratio of the moduli of various layers.

- Types of flexible pavements:
  - Thin HMA over aggregate base
  - Thick HMA over aggregate base
  - Full depth HMA
  - HMA over stabilized base
  - HMA over existing HMA or PCC (similar to composite pavement concept)
Flexible Pavement Model

Surface: $E_1$, $\mu_1$

Base: $E_2$, $\mu_2$

Subgrade: $E_3$, $\mu_3$
Stresses and Strains in Flexible Pavements

- **Function of the following:**
  - Material properties of each layer
  - Thickness of each layer
  - Loading conditions
  - Environmental impact

- **Pavement responses generally of interest:**
  - Surface deflection (represents surface rutting).
  - Horizontal tensile strain at bottom of AC layer (controls bottom up fatigue cracking).
  - Vertical compressive strain on top of intermediate layer (base or subbase rutting).
  - Vertical compressive strain on top of the subgrade (controls subgrade rutting).
Pavement Response Locations Used in Evaluating Load Effects

1. Pavement surface deflection
2. Horizontal tensile strain at bottom of bituminous layer
3. Vertical compressive strain at top of base
4. Vertical compressive strain at top of subgrade
One-Layer System
(Boussinesq Equations)

- The original elastic theory published by Boussinesq in 1885.
- For computing stresses and deflections in an elastic half-space material composed of homogeneous, isotropic, and linearly elastic material.
- Still widely used in soil mechanics and foundation design.

\[
\begin{align*}
\sigma_z &= \frac{P}{2\pi R^2} \left[ \frac{3z^2}{R^3} \right] \\
\sigma_r &= \frac{P}{2\pi R^2} \left[ \frac{3r^2z}{R^3} - (1-2\mu) \frac{R}{R+z} \right] \\
\sigma_\theta &= \frac{P}{2\pi R^2} (1-2\mu) \left[ \frac{R}{R+z} - \frac{z}{R} \right]
\end{align*}
\]

\[
\begin{align*}
u_z &= \frac{P(1+\mu)}{2\pi E} \left[ \frac{2(1-\mu)}{R} + \frac{z^2}{R^3} \right] \\
u_\theta &= \frac{P(1-\mu^2)}{\pi ER} \\
@ z = 0
\end{align*}
\]
Formulas for Calculating Stresses under a Circular Loaded Area

\[ \sigma_z = \sigma_0 \left[ 1 - \left( \frac{z}{\sqrt{a^2 + z^2}} \right)^3 \right] \]

\[ \sigma_r = \frac{\sigma_0}{2} \left[ 1 + 2\mu - \frac{2(1+\mu)z}{\sqrt{a^2 + z^2}} + \left( \frac{z}{\sqrt{a^2 + z^2}} \right)^3 \right] \]

\[ \sigma_t = \frac{\sigma_0}{2} \left[ 1 + 2\mu - \frac{2(1+\mu)z}{\sqrt{a^2 + z^2}} + \left( \frac{z}{\sqrt{a^2 + z^2}} \right)^3 \right] \]

\[ d_z = \int_0^\infty \varepsilon_z \, dz = \frac{\sigma_0 a (1+\mu)}{E} \left\{ \frac{1}{\sqrt{1+(z/a)^2}} + (1-2\mu) \left[ \sqrt{1+(z/a)^2} - (z/a) \right] \right\} \]

\[ d_o = \frac{2\sigma_o a (1-\mu^2)}{E} \quad @ \quad z = 0 \]
Generalized Hook’s Law
Elasticity Equations for the Calculation of Strains

\[ \varepsilon_z = \frac{1}{E} \left( \sigma_z - \mu(\sigma_r + \sigma_t) \right) \]

\[ \varepsilon_r = \frac{1}{E} \left( \sigma_r - \mu(\sigma_t + \sigma_z) \right) \]

\[ \varepsilon_t = \frac{1}{E} \left( \sigma_t - \mu(\sigma_r + \sigma_z) \right) \]
Calculation of the Strains

\[ \varepsilon_z = \frac{\sigma_0 (1 + \mu)}{E} \left\{ 1 - 2\mu - \frac{2\mu (z/a)}{\sqrt{1 + (z/a)^2}} - \left[ \frac{z/a}{\sqrt{1 + (z/a)^2}} \right]^3 \right\} \]

\[ \varepsilon_r = \frac{\sigma_0 (1 + \mu)}{2E} \left\{ 1 - 2\mu - \frac{2(1-\mu) (z/a)}{\sqrt{1 + (z/a)^2}} + \left[ \frac{z/a}{\sqrt{1 + (z/a)^2}} \right]^3 \right\} \]

\[ \varepsilon_t = \frac{\sigma_0 (1 + \mu)}{2E} \left\{ 1 - 2\mu - \frac{2(1-\mu) (z/a)}{\sqrt{1 + (z/a)^2}} + \left[ \frac{z/a}{\sqrt{1 + (z/a)^2}} \right]^3 \right\} \]

The equations are only valid along the load centerline.
Deflection Calculations

Calculation of the deflections at different depths along the load centerline:

\[ d_z = \int \frac{\sigma_0 a (1 + \mu)}{E} \left\{ \frac{1}{\sqrt{1 + (z/a)^2}} + (1 - 2\mu) \left[ \sqrt{1 + (z/a)^2} - \frac{z}{a} \right] \right\} dz \]

Surface deflections along the load centerline (@ \( z=0 \)):

\[ d_o = \frac{2\sigma_0 a (1 - \mu^2)}{E} \]
Nature of Responses under Flexible and Rigid Plates

**Flexible plate:**
- Uniform Contact Pressure
- Variable Deflection Profile

**Rigid Plate plate:**
- Non-Uniform Contact Pressure
- Equal Deflection

![Flexible Plate](image)

$$q(r) = \frac{qa}{2(a^2 - r^2)^{0.5}}$$

![Rigid Plate](image)
Stresses under the Rigid Plate

**Reminder:** Contract stresses are non-uniform under a rigid plate.

\[ \sigma_z = \frac{\sigma_0}{2\sqrt{1-(r/a)^2}} \]

\( \sigma_0 \) is the average pressure acting on the rigid plate (such as concrete slab).

\[ \sigma_0 = \frac{P}{\pi a^2} \]
Deflection under the Rigid Plate

*Reminder:* Deflections are equal under a rigid plate.

\[ d_0 = \frac{\pi \sigma_0 a (1 - \mu^2)}{2E} = \frac{P (1 - \mu^2)}{2Ea} = u_r \]

\[ \sigma_0 = \frac{P}{\pi a^2} \]

\( \sigma_0 \) is the average pressure acting on the rigid plate (such as concrete slab).
Comparison of Deflections at the Surface
Rigid vs. Flexible Plate

\[ d_o = \frac{2\sigma_o a (1-\mu^2)}{E} \]

**Flexible Plate**

\[ d_o = \frac{\pi \sigma_o a (1-\mu^2)}{2E} \]

**Rigid Plate**

\[
\frac{d_o \text{flexible}}{d_o \text{rigid}} = \frac{2 \frac{\sigma_o a (1-\mu^2)}{E}}{\frac{\pi \sigma_o a (1-\mu^2)}{2E}} = \frac{4}{\pi} = 1.27
\Rightarrow \frac{d_o \text{rigid}}{d_o \text{flexible}} = \frac{\pi}{4} = 0.79
\]

The deflection under a rigid plate is 79% of that under a flexible plate.
Foster and Ahlvin (1954) developed charts for computing vertical, tangential and radial stresses. The charts were developed for $\mu = 0.5$.

This work was subsequently refined by Ahlvin and Ulery (1962) allowing for evaluation of stresses and strains at any point in the homogenous mass for any $\mu$.

Due to symmetry, there are only three normal stresses, $\sigma_z$, $\sigma_r$ and $\sigma_t$ and one shear stress $\tau_{rz}$.

One-layer theory can be applied as an approximation for a conventional flexible pavement with granular base/subbase with a thin asphaltic layer on a stiff subgrade comparable to the base/subbase. (i.e., $E_1/E_2 \approx 1$).

The deflection that occurs within the pavement layers ($\Delta_p$) is neglected and therefore, the pavement surface deflection ($\Delta_T$) is equal to the deflection on the top of subgrade ($\Delta_s$):

$$\Delta_T = \Delta_p + \Delta_s, \quad \Delta_p = 0,$$

therefore $\Delta_T = \Delta_s$.
Charts for One Layer Solutions
(after Foster and Ahlvin, 1954)

Figure 2.2 Vertical stresses due to circular loading. (After Foster and Ahlvin (1954).)
Charts for One Layer Solutions
(after Foster and Ahlvin, 1954)

Figure 2.5 Shear stresses due to circular loading. (After Foster and Ahlvin (1954).)
Charts for One Layer Solutions (after Foster and Ahlvin, 1954)

Figure 2.6 Vertical deflections due circular loading. (After Foster and Ahlvin (1954).)
Charts for One Layer Solutions
(after Foster and Ahlvin, 1954)
<table>
<thead>
<tr>
<th>Parameter</th>
<th>General Case</th>
<th>Special Case ($\mu = 0.5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical stress</td>
<td>$\sigma_z = p[A + B]$</td>
<td>(same)</td>
</tr>
<tr>
<td>Radial horizontal stress</td>
<td>$\sigma_r = p[2\mu A + C + (1 - 2\mu)F]$</td>
<td>$\sigma_r = p[A + C]$</td>
</tr>
<tr>
<td>Tangential horizontal stress</td>
<td>$\sigma_t = p[2\mu A - D + (1 - 2\mu)E]$</td>
<td>$\sigma_t = p[A - D]$</td>
</tr>
<tr>
<td>Vertical radial shear stress</td>
<td>$\tau_{rz} = \tau_{rt} = pg$</td>
<td>(same)</td>
</tr>
<tr>
<td>Vertical strain</td>
<td>$\varepsilon_z = \frac{p(1 + \mu)}{E_1} [(1 - 2\mu)A + B]$</td>
<td>$\varepsilon_z = \frac{1.5p}{E_1} B$</td>
</tr>
<tr>
<td>Radial horizontal strain</td>
<td>$\varepsilon_r = \frac{p(1 + \mu)}{E_1} [(1 - 2\mu)F + C]$</td>
<td>$\varepsilon_r = \frac{1.5p}{E_1} C$</td>
</tr>
<tr>
<td>Tangential horizontal strain</td>
<td>$\varepsilon_t = \frac{p(1 + \mu)}{E_1} [(1 - 2\mu)E - D)]$</td>
<td>$\varepsilon_t = -\frac{1.5p}{E_1} D$</td>
</tr>
<tr>
<td>Vertical deflection</td>
<td>$\Delta_z = \frac{p(1 + \mu)\mu}{E_1} \left[\frac{2}{a} A + (1 - \mu)H\right]$</td>
<td>$\Delta_z = \frac{1.5pa}{E_3} \left(\frac{z}{a} + \frac{H}{2}\right)$</td>
</tr>
<tr>
<td>Bulk stress</td>
<td>$\theta = \sigma_z + \sigma_r + \sigma_t$</td>
<td></td>
</tr>
<tr>
<td>Bulk strain</td>
<td>$\varepsilon_{\theta} = \varepsilon_z + \varepsilon_r + \varepsilon_t$</td>
<td></td>
</tr>
<tr>
<td>Vertical tangential shear stress</td>
<td>$\tau_{xz} = \tau_{zt} = 0$ $\therefore [\sigma_z(\varepsilon_t) is principal stress (strain)]$</td>
<td></td>
</tr>
<tr>
<td>Principal stresses</td>
<td>$\sigma_{1,2,3} = (\sigma_z + \sigma_r) \pm \sqrt{(\sigma_z - \sigma_r)^2 + (2\tau_{xz})^2}$</td>
<td></td>
</tr>
<tr>
<td>Maximum shear stress</td>
<td>$\tau_{max} = \frac{\sigma_z - \sigma_r}{2}$</td>
<td></td>
</tr>
</tbody>
</table>
Burmister’s Theory of Two-Layer Systems

- As we discussed in the first lecture, one of the primary functions of the pavements is to protect the subgrade.

- Burmister (1958) obtained solutions for two-layer problem by using strain continuity equations.

- Based on Burmister’s solutions, vertical stresses are greatly influenced by the modular ratio (i.e., $E_1/E_2$).

- Vertical stress decreases considerably with increase in modular ratio. For example, based on the plot provided in the next slide:
  - for $z/a=1$ and $E_1/E_2 = 1$, $\sigma_z$ at interface = 68% of contact pressure.
  - for $z/a=1$ and $E_1/E_2 = 100$, $\sigma_z$ at interface = 8% of contact pressure.
Burmister’s Solutions for Vertical Stresses
Burmister extended the one-layer solutions to two and three layers in 1958.

He assumed layers have full frictional contact at the interface and $v=0.5$.

Equations and graphs are typically used to calculate the responses under the load.

The deflection under flexible plate ($w_0$) for two layer system can be calculated from the following equation:

$$w_0 = \frac{1.5 Pa}{E_2} F_2$$

P: Load distributed over a circular plate  
a: Radius of flexible plate  
$E_1$: Modulus of elasticity for the surface layer  
$E_2$: Modulus of elasticity of the subgrade  
$F_2$: Deflection coefficient
Burmister’s Solutions for Surface Deformations

The deflection coefficient ($F_2$) can be estimated from the following graph:
Burmister’s Solutions for Vertical Interface Stresses for Two Layer Systems
Three Layer Systems

- Fox and Acum developed closed form solutions for boundary stresses in the center of a circular uniformly loaded area.
- They assumed Passion's ratio of 0.5 for all layers.
- Later Jones and Peattie (1962) expanded the equations for three layer systems, they developed graphical solutions of responses based on the following parameters:

\[ K_1 = \frac{E_1}{E_2} \]
\[ K_2 = \frac{E_2}{E_3} \]
\[ A = \frac{a}{h_2} \]
\[ H = \frac{h_1}{h_2} \]

Schematic plot showing the locations of the pavement response solutions.
Influence of Layer Thickness on Vertical Stress Distributions ($\sigma_z$)

$K_1 = E_1/E_2 = 20$
$K_2 = E_2/E_3 = 20$

$A = a/h_2$
$H = h_1/h_2$

Vertical stress increases as the asphalt layer becomes thinner
Influence of Modular Ratio $K_1$ on Vertical Stress Distributions ($\sigma_z$)

$K_1 = E_1/E_2$

Vertical stress at the bottom of surface layer decreases as top layer stiffness increases

Vertical stress at the top of subgrade decreases slightly as asphalt layer stiffness increases

$K_1 = E_1/E_2$

$K_2 = E_2/E_3 = 10$

$A = a/h_2 = 1$

$H = h_1/h_2 = 1/4$
Influence of Modular Ratio $K_2$ on Vertical Stress Distributions ($\sigma_z$)

$K_1 = E_1/E_2 = 20$
$K_2 = E_2/E_3$
$A = a/h_2 = 1$
$H = h_1/h_2 = 1/4$

Vertical stress at the top of subgrade decreases significantly as base stiffness increases.
Typical Distribution of the Shear Stresses in Multi-Layer Systems

- Influence of modulus ratio $K_1 (= E_1/E_2)$ on the distribution of shear stresses.

- $K_1 = 1$ represents the Boussinesq's solution for single layer system ($E_1 = E_2$).

- Notice that the increasing modular ratios results in the maximum shear stress in the top layer, however the shear stress at interface is inversely related to the modular ratios.
Typical Distribution of the Shear Stresses in Multi-Layer Systems

- Influence of layer thickness ($a/h_1$) on the distribution of shear stresses.
- Notice the effect of nonlinearity in shear stresses for very thin surface layers (higher values of $a/h_1$).
- Both nonlinearity and magnitude of the shear stresses increase with reduction in surface layer thickness. This is why very thin asphalt pavements are prone to develop shear deformations. There is no control for shear deformation of flexible pavements in any design guide.
Odemark’s Method of Equivalent Thickness

Odemark developed a method to transform a system consisted of several layers with different stiffness properties into one single layer with one modulus value. Elastic-half space equations such as Boussinesq solutions can be used to calculate the responses under the wheel load.
Odemark’s Method of Equivalent Thickness

\[
\frac{h_e^3 E_2}{1-\mu_2^2} = \frac{h_1^3 E_1}{1-\mu_1^2}
\]

\[
h_e = h_1 \sqrt[3]{\frac{E_1}{E_2} \left( \frac{1-\mu_2^2}{1-\mu_1^2} \right)}
\]

Assuming the Passions' ratio of the two layers to be the same we have:

\[
h_e = h_1 \sqrt[3]{\frac{E_1}{E_2}}
\]
Method of Equivalent Thicknesses (Odemark’s General Equation)

\[
h_{ei} = h_i \times \frac{E_i}{E_{i+1}} \times \frac{1 - \mu_{i+1}^2}{(1 - \mu_i^2)}
\]

- \( h_{ei} \): Calculated equivalent thickness for \( i^{th} \) layer
- \( h_i \): Layer thickness for \( i^{th} \) layer
- \( E_i \): Modulus for \( i^{th} \) layer
- \( E_{i+1} \): Modulus for \((i+1)^{th}\) layer
- \( \mu_i \): Poisson’s ratio for \( i^{th} \) layer
- \( \mu_{i+1} \): Poisson’s ratio for \((i+1)^{th}\) layer
Odemark Equivalent Layer - Example

\[ h_e = h_1 \sqrt[3]{\frac{E_1}{E_2}} \]

\[ h_{e,2} = 0.8 \times 24.9 \times \sqrt[3]{5} = 34.1'' \]

\[ h_2 = 24.9'' \]

\[ h_1 = 6'' \]

\[ E_1 = 500 \text{ ksi} \]

\[ E_2 = 50 \text{ ksi} \]

\[ h_3 = 1.0 \times 6'' \times \sqrt[3]{10} = 12.9'' \]

\[ E_3 = 10 \text{ ksi} \]

\[ \sigma_0 = 90 \text{ psi} \]
Multi-Layer Systems Responses

- **Computer Programs**
  - WINJULIA
  - KENLAYER
  - ELSYM5
  - LEAP2
  - EVERSTRS

- **Typical Input**
  - Material properties: modulus (E) and Poisson's Ratio (v).
  - Layer thicknesses.
  - Loading conditions: magnitude of axle load, gear configurations, contact radius (tire footprint), or contact pressure.
  - Slip between layers (fully bonded or partially bonded layers).
Limitations of Layered Elastic Models (LEM)

- Effects of wheel loads applied close to cracks or edges (or joints in rigid pavements) require asymmetry - not available in LEM.

- Information on slip generally not available – influence can be dramatic – (BISAR program accommodates for the slip and inter-layer shear).

- Geostatic stresses are neglected.

- Vertical and lateral variation in dynamic or resilient modulus cannot be accounted for in LEM (Anisotropy).

- Stresses and strains calculated in unbound materials can be unreasonable – such as unrealistically high tensile stresses at the bottom of UAB due to stress sensitivity and anisotropy of the materials. (more discussion on this will be presented in the lecture on the characterization of unbound granular materials).
Layered elastic method doesn’t account for dynamic nature of the wheel load.

LEM represent tire-pavement contact as uniformly loaded, circular area while research demonstrates that actual contact area shape varies with loading and tire specifications.
Determination of the Tire Footprint
Vertical Stress Distribution under Dual Wheel Load

- Inflation Pressure = 520 kPa
- Applied Vertical Load (H/V) = 40 kN
- Average Wheel speed = 0.34 m/s
- Max Stress = 0.849 MPa

- Measured Vertical Load (CS) = 24 kN
- Measured Vertical Load (TS) = 23 kN
- Measured Vertical Total Load = 46.93 kN
Super Position of Wheel Loads
Example Pavement (6” Base)

\[ E_1 = 500,000 \text{ psi (3,450 MPa)} \]
\[ E_2 = 25,000 \text{ psi (172 MPa)} \]
\[ E_3 = 7,500 \text{ psi (52 MPa)} \]
Example Pavement (10” Base)

\[ E_1 = 500,000 \text{ psi (3,450 MPa)} \]
\[ E_2 = 25,000 \text{ psi (172 MPa)} \]
\[ E_3 = 7,500 \text{ psi (52 MPa)} \]
Example Pavement (14” Base)

\[ E_1 = 500,000 \text{ psi (3,450 MPa)} \]
\[ E_2 = 25,000 \text{ psi (172 MPa)} \]
\[ E_3 = 7,500 \text{ psi (52 MPa)} \]
Vertical Stress Distributions

Vertical Stress, psi

Depth, in

- Soft Subgrade (4 ksi)
- Stiff Subgrade (12 ksi)